

Empirical Mantissa Distributions of Pulsars

Lijing Shao and Bo-Qiang Ma

*School of Physics and State Key Laboratory of Nuclear Physics and Technology,
Peking University, Beijing 100871, China*

Abstract

The occurrence of digits one through nine as the leftmost nonzero digit of numbers from real world sources is often not uniformly distributed, but instead, is distributed according to a logarithmic law, known as Benford's law. Here, we investigate systematically the mantissa distributions of some pulsar quantities, and find that for most quantities their first digits conform to this law. However, the barycentric period shows significant deviation from the usual distribution, but satisfies a generalized Benford's law roughly. Therefore pulsars can serve as an ideal assemblage to study the first digit distributions of real world data, and the observations can be used to constrain theoretical models of pulsar behavior.

Key words: pulsar, mantissa distribution, Benford's law, first digit law

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Email address: mabq@phy.pku.edu.cn (Bo-Qiang Ma).

1 Introduction

Pulsars, celestial lighthouses in the sky, are excellent natural laboratories for the research of fundamental properties of matter under the circumstances of strong gravity, strong magnetic field, high density, and extremely relativistic condition. Pulsar physics has been a forefront field of both astronomy and physics for more than 40 years since its discovery. Though exciting progresses in this domain have significantly enlarged our knowledge of astronomical environment and physical processes, there still remain outstanding problems [1,2]. Nowadays, due to the efforts made by modern observational instruments covering over various wavelength scopes, *e.g.*, ROSAT, BeppoSAX, Chandra, XMM-Newton, HST, and VLT, people accumulated a sea of pulsar data, and the data are accreting significantly all these days. Hence data analysis and statistical synthesis become a vital step to characterize pulsar properties and reveal their inner regularities [3,4].

In this paper, we perform a systematic investigation of the mantissa distributions of pulsars for the first time. The mantissa $m \in (-1, -0.1] \cup [0.1, 1)$ is the significant part of a floating-point number x , defined as $x = m \times 10^n$, where n is an integer. In this paper, if not noted explicitly, we always postulate that the numbers are positive for succinct statement.

One might presume that the mantissas of any randomly chosen data set are approximately uniformly distributed, but that is not the case in real world. Instead, as stated by Newcomb [5], *“the law of probability of the occurrence of numbers is such that all mantissae of their logarithms are equally likely”*. Subsequently, it leads to the conclusion that the first significant digit, i.e., 1, 2, ..., 9, of mantissa is logarithmically distributed, where the number 1 appears almost seven times more often than that of the number 9. The probability of the occurrence of the first digit can be expressed in an analytical formula, called Benford’s law [6] after the name of its second discoverer,

$$P(k) = \log_{10}\left(1 + \frac{1}{k}\right), \quad k = 1, 2, \dots, 9 \quad (1)$$

where $P(k)$ is the probability of a number having the first digit k .

Empirically, the areas of lakes, the lengths of rivers, arabic numbers on the front page of a newspaper [6], physical constants and distributions [7,8], the stock market indices [9], file sizes in a personal computer [10], survival distributions [11], widths of hadrons [12], even dynamical systems [13,14,15], conform to the peculiar law well. Nevertheless, there also exist other types of data, *e.g.*, lottery and telephone numbers, which do not obey the law. Unfortunately, there is no a priori criteria yet to judge which type a data set belongs

to. In practice, the law is already applicable in distinguishing and ascertaining fraud in taxing and accounting [16,17,18,19], and speeding up calculation and minimizing expected storage space in computer science [20,21,22].

Since its second discovery in 1938, many attempts have been tried to explain the underlying reason for Benford’s law. For theoretical reviews, see papers written by Raimi [23,24,25] and Hill [26,27,28,29,30]. Nowadays, many breakthrough points have been achieved in this domain, though, there still lacks a universally accepted final answer. In mathematics, Benford’s law is the only digit law that is scale-invariant [31], which means that the law does not depend on any particular choice of units, discovered by Pinkham [32]. Also Benford’s law is base-invariant [26,27,28], which means that it is independent of the base d you use. In the binary system ($d=2$), octal system ($d=8$), or other base system, the data, as well as in the decimal system ($d=10$), all fit the general first digit law, $P(k) = \log_d(1 + 1/k)$, $k \in \{1, 2, \dots, d - 1\}$. Theoretically, Hill proved that “*scale-invariance implies base-invariance*” [26] and “*base-invariance implies Benford’s law*” [27] mathematically in the framework of probability theory. He also proved that random entries, picked from random distributions, form a sequence whose significant digit distribution approaches to Benford’s law [28].

Intending to uncover some regularities of pulsar data, and also to explore new domains of the digit law, we investigate the mantissa distributions of pulsar quantities systematically. We find that the first nonzero digit of mantissas displays unevenness according to the logarithmic law. The exceptions are the barycentric period and rotation frequency, which show significant deviations from Benford’s law. Further, we also discuss various properties of the digit law, and perform the generalized Benford’s law to data sets of barycentric period and rotation frequency of pulsars as well. Therefore the data of pulsars provide an ideal assemblage for further studies on the first digit law of the nature.

2 First digit distributions

We investigate the famous Australia Telescope National Facility (ATNF) pulsar catalogue¹, which is a widely used database, maintained by Manchester *et al.* [33]. Thanks to their exhaustive search of pulsar literatures, at least back to 1993, data from all papers announcing the discoveries of pulsars or giving improved parameters are entered into the catalogue database. As ATNF pulsar catalogue is an updating database, to avoid ambiguity, in this paper we make use of *Catalogue version 1.36*, including totally 1826 samples of pulsars.

¹ <http://www.atnf.csiro.au/research/pulsar/psrcat>

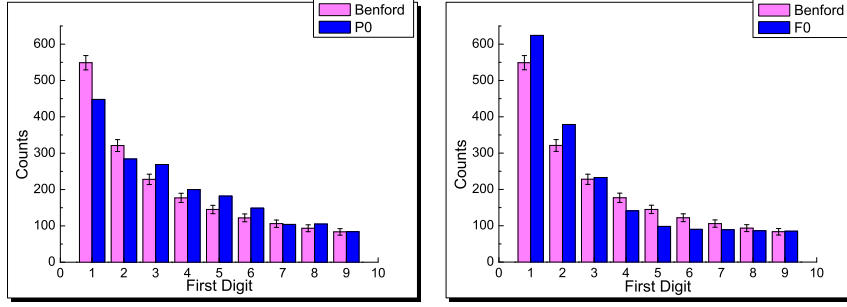


Fig. 1. Comparisons of Benford’s law and the distributions of the first digit of the barycentric period (left) and rotation frequency (right) of pulsars.

Table 1

The first digit distributions of the period, frequency, spin down ages, and their time derivatives of pulsars.

Physical Quantity	Notation	Data points	$\chi^2(8)$	p -value
Barycentric period of the pulsar (s)	P0	1825	50.552	0.0001
Barycentric rotation frequency (Hz)	F0	1825	54.577	0.0001
Time derivative of barycentric period (dimensionless)	P1	1695	4.497	0.8097
Period derivative corrected for proper motion effect	P1.i	219	9.502	0.3017
Time derivative of barycentric rotation frequency (s^{-2})	F1	1695	7.539	0.4797
Second time derivative of barycentric rotation frequency (s^{-3})	F2	395	3.020	0.9331
Spin down age (yr) [$\tau = P/(2\dot{P})$]	Age	1664	3.721	0.8814
Spin down age from P1.i (yr)	Age.i	219	7.078	0.5282

However, not every pulsar item has complete information, e.g., one of 1826 pulsars, J1911+00, in this database has no information on barycentric period hence rotation frequency. We cover every presented nonzero number without bias. The notations of physical quantities in the following text are the same as that from this catalogue.

The first digit distributions of the barycentric period in the unit of second, P0, and rotation frequency in the unit of Hertz, F0, of pulsars are illustrated in Figure 1 and Table 1. There are totally $N = 1825$ available samples as mentioned. The theoretical predictions in the figure are the expected number, $N_{\text{Ben}} = N \log_{10}(1 + 1/k)$, together with the root mean square error evaluated

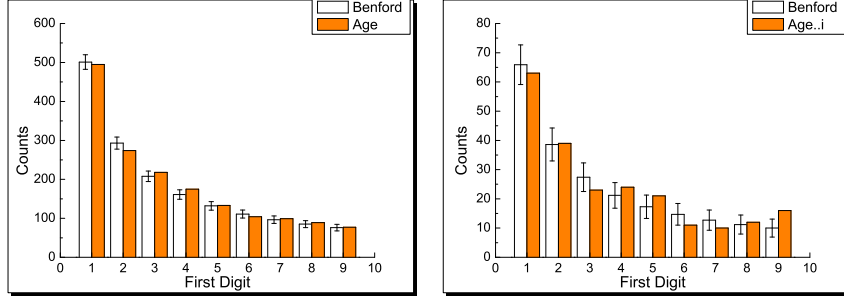


Fig. 2. Comparisons of Benford's law and the distributions of the first digit of the spin down age (left) and the spin down age from P1.i (right) of pulsars.

by the binomial distribution [34,35], $\Delta N = \sqrt{N P(k) (1 - P(k))}$. We use the Pearson χ^2 to estimate goodness of fit to the probability distribution,

$$\chi^2(n-1) = \sum_{i=1}^n \frac{(N_{\text{Obs}} - N_{\text{Ben}})^2}{N_{\text{Ben}}} \quad (2)$$

where N_{Obs} is the observational number and N_{Ben} is the theoretical prediction from Benford's law, and here in our question $n = 9$. In Eq. (2), the degree of freedom is $9 - 1 = 8$, and under the confidence level (CL) 95%, $\chi^2(8) = 15.507$.

From Figure 1, we see that they are roughly consistent but with considerable divergence from the Benford's law. The deviations are actually just one deviation, since period and frequency are inversely related. Pearson χ^2 in Table 1 are rather large, thus the χ^2 tests reject the null hypothesis H_0 : *the barycentric period and rotation frequency of pulsars fit Benford's law*. Worthy to mention that, their p -values, which measure the probability of obtaining a test statistic at least as extreme as the one that is actually observed, are rather small accordingly. The lower the p -value, the less likely the null hypothesis. In our analysis, we reject a null hypothesis if its p -value is less than 0.05 (equivalent to under CL 95%). The anomaly of period and frequency from Benford's law will be discussed and handled with the generalized Benford's law latter in the paper.

However, as depicted in Figure 2 and Table 1, comparisons of Benford's law and the distributions of the first digit of the spin down ages of pulsars, Age and Age.i, are both very impressive. The ages are estimated as $\tau = P/(2\dot{P})$, where $P = P_0$, and $\dot{P} = \dot{P}_1$ for Age while $\dot{P} = \dot{P}_{1.i}$ for Age.i. As listed in Table 1, \dot{P}_1 and $\dot{P}_{1.i}$ are observational time derivative of barycentric period and the period derivative corrected for proper motion effect respectively. The χ^2 strongly supports the null hypothesis H_0 : *the spin down ages of pulsars*

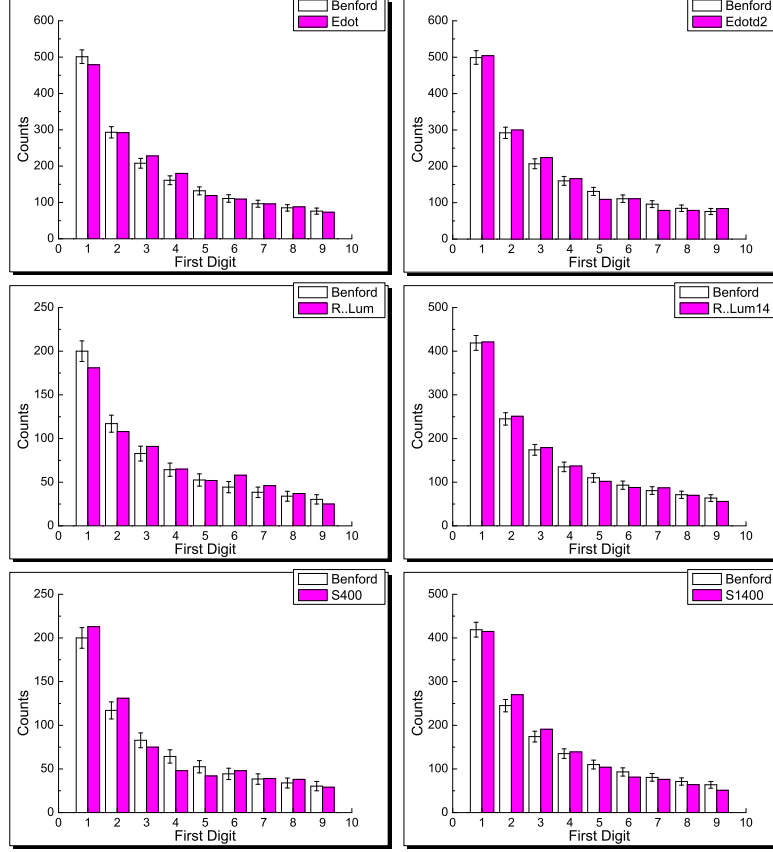


Fig. 3. Comparisons of Benford’s law and the distributions of the first digit of the power quantities of pulsars. From top left to bottom right, they are spin down energy loss rate, energy flux at the Sun, radio luminosity at 400 MHz, radio luminosity at 1400 MHz, mean flux density at 400 MHz, and mean flux density at 1400 MHz.

follow Benford’s law. Torres *et al.* [10] discussed the possibility of transformation from non-Benford data sets to Benford ones. They started from evenly distributed random numbers, and found that the group of numbers created by random multiplication between them reveals uneven property and fulfils the logarithmic law. Correspondingly, we here give an empirical demonstration from a non-Benford set (the barycentric periods) and a Benford set (time derivatives of barycentric period) to a Benford one (the spin down ages).

The first digit distributions of power quantities of pulsars are depicted in Figure 3 and Table 2. In Figure 3, from top left to bottom right, they are spin down energy loss rate, energy flux at the Sun, radio luminosity at 400 MHz, radio luminosity at 1400 MHz, mean flux density at 400 MHz, and mean flux density at 1400 MHz. The χ^2 tests for all the power quantities are well supportive to H_0 : *the power quantities of pulsars fit Benford’s law.*

Table 2

The first digit distributions of the power quantities of pulsars.

Physical Quantity	Notation	Data points	$\chi^2(8)$	p -value
Spin down energy loss rate (ergs/s)	Edot	1664	6.601	0.5802
Energy flux at the Sun (ergs/kpc ² /s)	Edotd2	1656	9.938	0.2694
Radio luminosity at 400 MHz (mJy kpc ²)	R_Lum	663	10.083	0.2592
Radio luminosity at 1400 MHz (mJy kpc ²)	R_Lum14	1391	2.673	0.9532
Mean flux density at 400 MHz (mJy)	S400	663	10.446	0.2351
Mean flux density at 1400 MHz (mJy)	S1400	1391	9.855	0.2754

Table 3

The first digit distributions of the kinematic quantities of pulsars.

Physical Quantity	Notation	Data points	$\chi^2(8)$	p -value
Proper motion in declination (mas/yr)	PMDec	178	4.840	0.7745
Proper motion in right ascension (mas/yr)	PMRA	178	8.421	0.3935
Proper motion in ecliptic latitude (mas/yr)	PMElat	39	13.057	0.1099
Proper motion in ecliptic longitude (mas/yr)	PMElong	83	11.695	0.1653
Total proper motion (mas/yr)	PMTot	220	7.527	0.4810
Transverse velocity – based on DIST (km/s)	VTrans	219	11.855	0.1578

Likewise, in Figure 4 and Table 3, we present the first digit distributions of kinematic quantities of pulsars. In Figure 4, there are proper motion in declination, proper motion in right ascension, proper motion in ecliptic latitude, proper motion in ecliptic longitude, total proper motion, and the transverse velocity based on the best estimated pulsar distance, from top left to bottom right. The χ^2 tests for all kinematic quantities, the total proper motion as well as components of the total motion, favor the null hypothesis H_0 : *the movements of pulsars conform to Benford’s law*.

As mentioned above, not all data sets respect Benford’s law. Especially, artificial numbers often diverge from the logarithmic distribution, and this is the very reason why Benford’s law is applied in detecting number frauds. Data

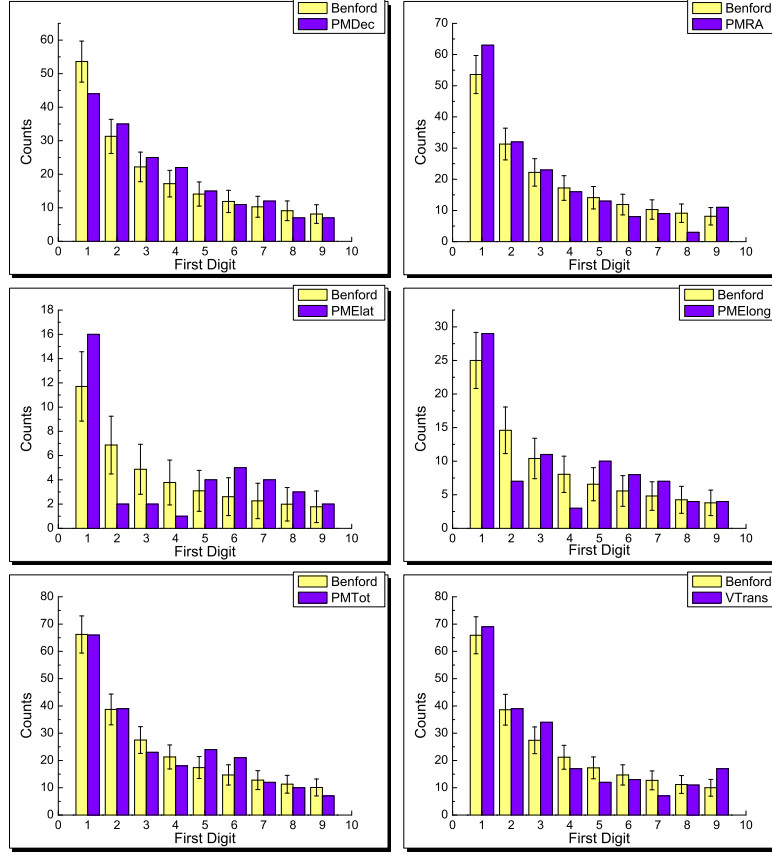


Fig. 4. Comparisons of Benford’s law and the distributions of the first digit of the kinematic quantities of pulsars. From top left to bottom right, they are proper motion in declination, proper motion in right ascension, proper motion in ecliptic latitude, proper motion in ecliptic longitude, total proper motion, and the transverse velocity based on the best estimated pulsar distance.

Table 4

The first digit distributions of the expected minimum companion mass and median companion mass of pulsars.

Physical Quantity	Notation	Data points	$\chi^2(8)$	p -value
Minimum companion mass assuming $i=90$ degrees and neutron star mass is $1.35 M_{\odot}$	MinMass	140	31.331	0.0001
Median companion mass assuming $i=60$ degrees	MedMass	140	32.781	0.0001

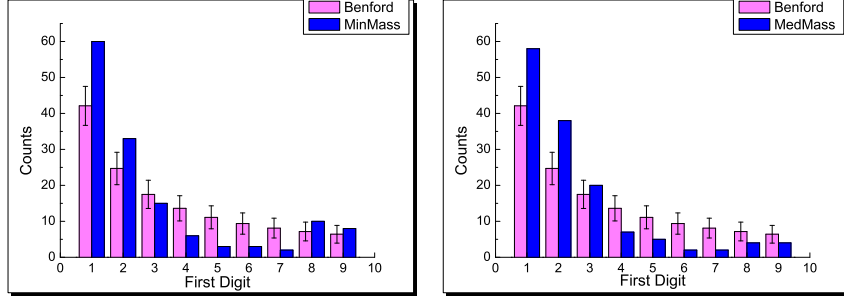


Fig. 5. Comparisons of Benford’s law and the distributions of the first digit of the expected minimum companion mass (left) and median companion mass (right) of pulsars.

sets that are arbitrary and contain restrictions usually do not comply with the peculiar law, in contrast, data sets measured from the real world are more likely to obey the law. The first digit distributions of the expected companion minimum mass, MinMass, and median companion mass, MedMass, of pulsars are illustrated in Figure 5 and Table 4. MinMass is the minimum companion mass calculated by assuming $i = 90$ degrees and neutron star mass is $1.35 M_{\odot}$, and MedMass is the median companion mass assuming $i = 60$ degrees, where i denotes the orbital obliquity. Apart from the feasibility of the rough assumptions, the expected minimum mass and median mass are neither natural numbers by measurement, and the physics of these companion masses also places constraints on the range of values, making it difficult for their distributions to be logarithmically uniform, thus the departures from the logarithmic distribution with rather large χ^2 for MinMass and MedMass are expected. However, in general, it still lacks universal justification for all examples.

As we can see at the beginning of this section, the barycentric period and rotation frequency are not good to follow Benford’s law, hence it raises an interesting question: do the time derivatives of these non-Benford data sets follow the significant digit law? In Figure 6, we present an empirical positive answer to the above question. From Table 1, we can see that the Pearson χ^2 for time derivative of barycentric period, period derivative corrected for proper motion effect, time derivative of barycentric rotation frequency, and second time derivative of barycentric rotation frequency, drive the null hypothesis H_0 : *the time derivatives of period and frequency of pulsars obey Benford’s law*, to be accepted.

Furthermore, we investigate the positive and negative numbers separately for this case, and this has not been discussed explicitly in other literatures. We find that the Pearson χ^2 for the 1664 positive numbers and 31 negative ones of the time derivative of barycentric period of pulsars are 5.364 and 9.977, while

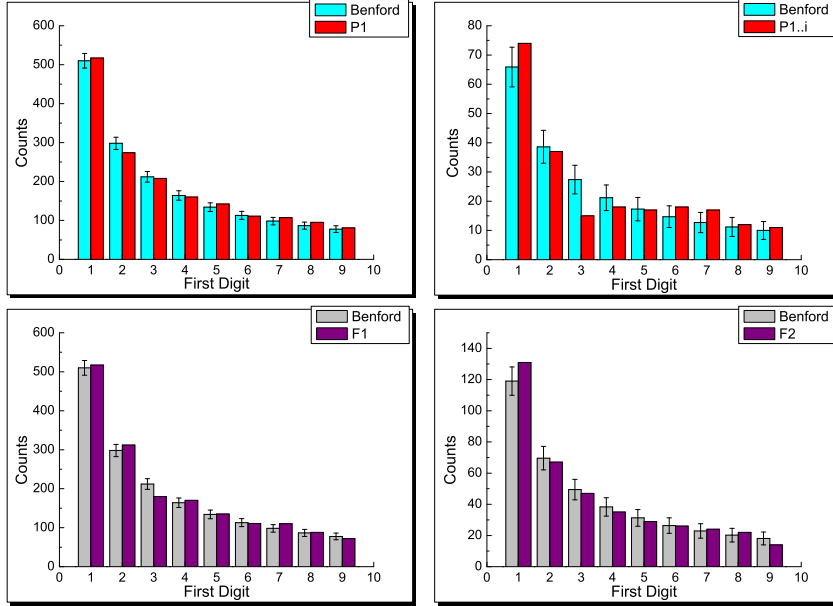


Fig. 6. Comparisons of Benford's law and the distributions of the first digit of the time derivatives of period and frequency of pulsars. From top left to bottom right, they are time derivative of barycentric period, period derivative corrected for proper motion effect, time derivative of barycentric rotation frequency, and second time derivative of barycentric rotation frequency.

for the time derivative of barycentric rotation frequency, 31 positive and 1664 negative numbers, χ^2 are 2.813 and 7.590, respectively. Thus we hint at the conclusion that the negative numbers, as well as positive ones, empirically apply to Benford's law. It is a challenge why the nature spontaneously organizes numbers into such a fantastic global order.

3 Generalized Benford's law

Pietronero *et al.* [36] provided a new insight, suggesting that a process or an object $m(t)$ with its time evolution governed by multiplicative fluctuations generates Benford's law naturally, and they used stockmarket as a convictive example. The main idea is that $m(t + \delta t) = r(t) \times m(t)$, where $r(t)$ is a random variable. After treating $\log r(t)$ as a new random variable, it is a Brownian process $\log m(t + \delta t) = \log r(t) + \log m(t)$ in the logarithmic space. Utilizing the central limit theorem in a large sample, $\log m(t)$ becomes uniformly dis-

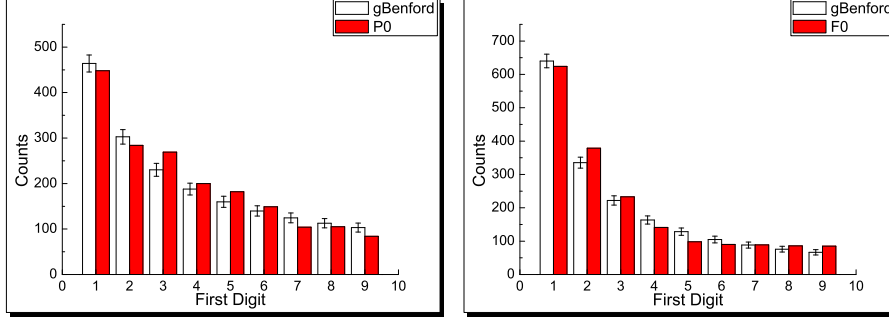


Fig. 7. Comparisons of the generalized Benford's law and the distributions of the first digit of the barycentric period (left) and rotation frequency (right) of pulsars. tributed. Thus,

$$P(k) = \frac{\int_k^{k+1} d \log m(t)}{\int_1^{10} d \log m(t)} = \frac{\int_k^{k+1} m^{-1} dm}{\int_1^{10} m^{-1} dm} = \log_{10}\left(1 + \frac{1}{k}\right), \quad (3)$$

which is exactly the formula of Benford's law given in Eq. (1). Later, the idea is well extended to affine processes $m(t + \delta t) = r(t) \times m(t) + r'(t)$ by Gottwald and Nicol [37] using techniques from ergodic theory.

From Eq. (3) above, we can see that the probability density of a number owning mantissa m is proportional to m^{-1} , a fact which was also gained by Lemons [38] in terms of a probabilistic model of partitioning a conserved quantity. Further, Pietronero *et al.* [36] generalized the probability density to be proportional to $m^{-\alpha}$, conserving the scale-invariant property, thus the probability to have the first digit k becomes $P_\alpha(k) = C \int_k^{k+1} m^{-\alpha} dm$, where C is a normalization factor. In the framework of generalized Benford's law, when $\alpha = 0$, it becomes the uniform distribution, while $\alpha = 1$ corresponds to Benford's law, and for increasing α , the first digit $k = 1$ appears more frequent, raising the unevenness of the digit distribution. Moreover, they analyzed the southern California catalogue for earthquake magnitudes with $\alpha \simeq 2$ and got rather good fitness. Recently, Luque and Lacase [39] utilized size-dependent exponent $\alpha(N)$ to analyse consequences of primes and Riemann zeta zeros.

Here we treat the non-Benford data sets of the barycentric period and rotation frequency of pulsars with the generalized Benford's law. The results are illustrated in Figure 7, where we adopt $\alpha = 0.8$ for barycentric period and $\alpha = 1.2$ for rotation frequency. Here the symmetry of exponents about 1 is expected, since two quantities are related reciprocally. Through comparisons between Figure 1 and Figure 7, we see clearly that the goodness of fit improves a lot. For the generalized form, the fitness χ^2 is deduced to $\chi^2(7) = 20.187$

for barycentric period and $\chi^2(7) = 25.569$ for rotation frequency. F -test gives $p = 0.0142$ and $p = 0.0258$ respectively, which rejects the null hypothesis that *no significant improvement is introduced after introducing α* . Hence the generalized Benford's law is favored for barycentric period and rotation frequency of pulsars, the χ^2 is still a bit large though.

Why and how do overwhelming majority of numbers develop into inverse probability density hence Benford's law? What is the inner regularity that distinguishes some generalized power-law anomaly from the specific logarithmic law? There still remain crucial challenges. Nevertheless, we can always benefit from regularities provided by the nature, and use Benford's law to inspect scientific and social data falsify, and also to test the validity of theoretical models. We hope to introduce pulsars data as a good assemblage for such kind of studies.

4 Summary

In this paper, we present systematic analysis on the first digit distributions of mantissas of most fundamental quantities of pulsars, including barycentric period and rotation frequency, together with their time derivatives, and many more. The results reveal obvious departures from the uniform distribution, and small digits are more prevalent than large ones according to a logarithmic formula, called Benford's law. However, not all data sets conform to it. Artificial and restricted data sets often diverge from the law, such as the expected minimum and median companion masses of pulsars in our study. Furthermore, the generalized Benford's law is applied to the barycentric period and rotation frequency of pulsars.

Since the first discovery of the peculiar significant digit law more than one hundred years ago, many crucial understandings of the inner regularities from mathematical viewpoint have been achieved. But there still remain big challenges to scientists to look into more profound physical reasons of such remarkable property of the nature. Thus the revelation of the hitherto unnoticed regularities in the pulsar data opens a new window to look into novel aspects of astronomical objects.

Acknowledgments

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